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1993 J. Phys. A: Math. Gen. 26 3597

(http://iopscience.iop.org/0305-4470/26/14/023)

J. Phys. A: Math. Gen. 26 (1993) 3597–3625. Printed in the UK

Soliton solutions of relativistic Hartree equations

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Received 14 September 1992, in final form 5 April 1993

Abstract. We study a model based on N scalar complex fields coupled to a scalar real field, where all fields are treated classically as c-numbers. The model describes a composite particle made up of N constituents with bare mass m_0 interacting both with each other and with themselves via the exchange of a particle of mass μ_0. The stationary states of the composite particle are described by relativistic Hartree equations. Since the self-interaction is included, the case of an elementary particle is a non-trivial special case of this model. Using an integral transform method we derive the exact ground-state solution and prove its local stability. The mass of the composite particle is calculated as the total energy in the rest frame. For the case of a massless exchange particle the mass formula is given in closed form. The mass, as a function of the coupling constant, possesses a well pronounced minimum for each value of μ_0/m_0, while the absolute minimum occurs at $\mu_0 = 0$.

1. Introduction and main results

In this paper we derive and study the exact ground-state solution in $3+1$ dimensions for a self-interacting system whose dynamics is governed by the relativistic Hartree equations. Our motivation is to study bound states formed solely by self-interaction. Such bound states (sometimes called non-topological solitons) offer a possibility of understanding the internal properties of particles, such as their masses, charges and magnetic moments. Self-interaction is a purely nonlinear and non-perturbative phenomenon which is neither well understood nor properly appreciated at present. However, we argue that in the quantum domain it is important and deserves close study. It may be the crucial missing element in our understanding of quantum phenomena. Self-interaction appears frequently in quantum field theories such as, for example, in quantum electrodynamics, where it leads to infinities. Intuitively, the reason why these infinities appear is that we are using a perturbation theory in which particles are associated with free fields such as plane waves or wave packets. The fact that one is able to cure the theory and remove the infinities in a self-consistent manner can be seen as an indication that the original theory, before the perturbation theory based on free fields is applied to it, is correct and should merely be treated differently. What is needed, perhaps, is an approach which is based on self-interaction bound states instead of free fields. An attempt at such an approach for the case of quantum electrodynamics will be presented in a forthcoming paper. In the present paper we attempt to present the self-interaction bound states in their own right, within the context of a classical field theory. In doing so we will concentrate on their internal properties in which they differ so much from free fields.

The following model is motivated by its simplicity, self-consistency and by the fact that it is connected to the large N_c limit of QCD. It is not intended to be realistic in the first

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place. Yet some of its elements, for instance relativistic invariance, dimension $3+1$ and the nonlinear interaction, are clearly realistic.

We consider a system of N complex scalar fields $\Psi_j(\boldsymbol{r}, t)$, $j = 1, \ldots, N$ and a real scalar field $\Phi(\boldsymbol{r}, t)$ with the Lagrangian

$$\mathcal{L} = -\sum_{j=1}^{N}(\partial^\nu \Psi_j^* \partial_\nu \Psi_j + m_0^2 \Psi_j^* \Psi_j - g\Phi^p(\Psi_j^*\Psi_j)^q) - \tfrac{1}{2}(\partial^\nu \Phi \partial_\nu \Phi + \mu_0^2 \Phi^2) \tag{1}$$

where $p = q = 1$, and the equations of motion

$$(\Box - m_0^2 + g\Phi(\boldsymbol{r}, t))\Psi_j(\boldsymbol{r}, t) = 0 \tag{2}$$

$$(\Box - \mu_0^2)\Phi(\boldsymbol{r}, t) = -g\sum_{i=1}^{N}\Psi_i^*(\boldsymbol{r}, t)\Psi_i(\boldsymbol{r}, t) \tag{3}$$

where $\Box \equiv \Delta - \partial^2/\partial t^2 \equiv \partial^\nu\partial_\nu$. All the fields are assumed to be classical, i.e. commuting. Notice that the interaction term in (1) is not positive definite and hence the stability question arises. In this paper only the local stability is investigated in full detail. Here it suffices to say that stable one-particle and many-particle bound states exist, with energies below the threshold to the continuum states, provided certain restrictions are placed on the coupling constant g and on μ_0/m_0. The fields Ψ_j are normalized according to

$$\int \mathrm{d}^3 r \left| \frac{\partial \Psi_j^*}{\partial t}\Psi_j - \Psi_j^* \frac{\partial \Psi_j}{\partial t} \right| = 1 \qquad j = 1, \ldots, N. \tag{4}$$

Notice that the left-hand side of (4) is a constant of motion. In the present context the 1 on the right-hand side of (4) is purely conventional, in fact we could equally well have chosen any other number. However, equations (2) and (3) allow for a rescaling of the fields, which can be chosen to restore the 1 in (4). The fact that Ψ_j are complex fields and hence are normalized according to (4) is an essential one since otherwise the model would contain $\mu_0 = m_0$, $\Psi = \Phi$ as a special case and then it would be identical to the standard Φ^3 model which is unstable even locally.

Equations (2)–(4) admit plane wave solutions with continuous energies $|E_k| \equiv k^2 + m_0^2 \geqslant m_0$

$$\Psi_j(\boldsymbol{r}, t) = \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}_j \cdot \boldsymbol{r} - \mathrm{i}E_{k_j} t}}{\sqrt{2|E_{k_j}|\Omega}} \qquad \Phi(\boldsymbol{r}, t) = \frac{g}{\mu_0^2}\frac{N}{2|E_{k_j}|\Omega} \qquad \Omega \to \infty \tag{5}$$

where Ω is an arbitrarily large normalization volume (a detailed discussion of plane wave solutions in the context of non-topological solitons is given in [1]). As will be shown below, the characteristic feature of these states is that for any N their total energy is always positive and larger than the corresponding bare mass Nm_0, i.e. $E_{\text{total}} \geqslant Nm_0$ and hence on the energy scale these states fill the continuum above Nm_0.

In contrast to the above continuum states there exist bound states with total energies $0 < E_{\text{total}} < Nm_0$. In the rest frame of the bound states the corresponding stationary state solutions of (2) and (3) are of the form $\Psi_j(\boldsymbol{r}, t) = \psi_j(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}E_j t}$ and $\Phi(\boldsymbol{r}, t) = \phi(\boldsymbol{r})/g$, where $|E_j| < m_0$. For these stationary fields the equations of motion (2), (3) and the normalization condition (4) become, respectively,

$$(\Delta - \gamma_j^2 + \phi(\boldsymbol{r}))\psi_j(\boldsymbol{r}) = 0 \tag{6}$$

$$(\Delta - \mu_0^2)\phi(\boldsymbol{r}) = -g^2\sum_{i=1}^{N}|\psi_i(\boldsymbol{r})|^2 \tag{7}$$

and

$$\int \mathrm{d}^3 r |\psi_j(r)|^2 = \frac{1}{2|E_j|} \qquad j = 1, \ldots, N \tag{8}$$

where $\gamma_j^2 = m_0^2 - E_j^2 \geqslant 0$. Once a solution is obtained, it can be Lorentz-boosted to an arbitrary inertial frame. Equations (6) and (7) must be supplemented with suitable boundary conditions (they are discussed below).

This system comprises a model of a composite particle made up of N spinless constituents of bare mass m_0 and the constituent energies $|E_j|$, which interact with each other and with themselves via the exchange of a particle of mass μ_0. We identify the physical mass M of the composite particle with its total energy E_{total}. The total energy follows in the usual way from the Lagrangian (1)

$$M \equiv E_{\text{total}} = \int \mathrm{d}^3 r \bigg(\sum_{j=1}^{N} [\nabla\psi_j^* \cdot \nabla\psi_j + (E_j^2 + m_0^2)\psi_j^*\psi_j - \phi\psi_j^*\psi_j] + \frac{1}{2g^2}[(\nabla\phi)^2 + \mu_0^2\phi^2] \bigg).$$

The gradient term $(\nabla\phi)^2$ can be eliminated using (7) and we obtain

$$M = \int \mathrm{d}^3 r \sum_{j=1}^{N} [\nabla\psi_j^* \cdot \nabla\psi_j + (E_j^2 + m_0^2)\psi_j^*\psi_j - \tfrac{1}{2}\phi\psi_j^*\psi_j].$$

Eliminating the remaining gradient term by the aid of (6) and using the normalization condition (8), we get

$$M = \sum_{j=1}^{N} \bigg(|E_j| + \tfrac{1}{2} \int \mathrm{d}^3 r\, \phi(r) |\psi_j(r)|^2 \bigg). \tag{9}$$

For the continuum states given in (5) the last term in (9) vanishes as $\Omega \to \infty$ and hence $E_{\text{total}} = \sum_{j=1}^{N} |E_j| \equiv \sum_{j=1}^{N} |E_{k_j}| \geqslant N m_0$. In the case of bound states the last term in (9) does not vanish and hence the total mass is not identical to the sum of the constituent energies $|E_j|$ (i.e. to the sum of the bare masses m_0 and the binding energies $m_0 - |E_j|$ of the constituents) but contains an additional term. Below we will show that for stationary states this term is strictly positive. While the binding energies can be interpreted as the energies which are gained by putting particles in an attractive potential, the last term in (9) represents the energy which must be spent to create the potential itself. In a consistent field theory these two sorts of energies are interdependent and always appear together.

In the case of $N = 1$ the mass formula (9) defines the mass of an elementary particle, which we denote by m. It will be shown below that for stable bound-state solutions of (6)–(7) we have $m \leqslant m_0$ and hence the ground states of particles are bound states and not the continuum states for which $m = m_0$. It is therefore not an assumption, but a necessity, that in this model the particles are described by self-interaction bound states rather than by plane waves or wave packets. The main aim of this paper is to solve (6)–(7) for the ground state and to determine the dependence of the total mass M and the size parameter r_0 on the coupling constant g, the bare mass m_0, the mass of the exchange particle μ_0 and the number of constituents N.

In a situation in which all of the constituent particles are in the same spherically symmetric state with $|E_j| \equiv |E| < m_0$ and $\psi_j(\boldsymbol{r}) \equiv \psi(\boldsymbol{r}) = \psi(r)/\sqrt{4\pi}$, the system of equations (6)–(7) simplifies to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \gamma^2 + \phi(r)\right)\psi(r) = 0 \tag{10}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \mu_0^2\right)\phi(r) = -\frac{g^2 N}{4\pi}|\psi(r)|^2 \tag{11}$$

where $\gamma^2 = m_0^2 - E^2$. Since we are mainly interested in the properties of the ground state, which is the lowest-energy solution of (10)–(11), we can restrict ourselves to this situation. For the mass we obtain

$$M = N\left(|E| + \frac{1}{2}\int \mathrm{d}^3r\,\phi(r)|\psi(r)|^2\right). \tag{12}$$

Contrary to the case of plane waves, the particles in the present model are extended. As a measure of their spatial extension, we define the size parameter

$$r_0 = 2|E|\int \mathrm{d}^3\boldsymbol{r}\, r|\psi(\boldsymbol{r})|^2 \tag{13}$$

where the factor $2|E|$ stems from the normalization condition (8).

The stationary states of the composite particle are described by Hartree equations. This follows from the fact that (7) possesses the well known solution

$$\phi(\boldsymbol{r}) \equiv 2mV(\boldsymbol{r}) = \frac{g^2}{4\pi}\sum_{i=1}^{N}\int \mathrm{d}^3\boldsymbol{r}'\,\frac{\mathrm{e}^{-\mu_0|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}|\psi_i(\boldsymbol{r}')|^2 \tag{14}$$

which, if inserted in (6), gives the usual Hartree equations with the effective potential $V(\boldsymbol{r})$. Notice that the self-interaction ($i = j$) is included in the Hartree equations and therefore the case of an elementary particle, where $N = 1$, is a non-trivial special case of this model. In addition notice that in the present model the Hartree equations are exact and not approximate as in the usual case. The model would be more realistic if the constituent particles were fermions. Yet, despite this shortcoming, it is useful. For instance, it was shown in [2] that baryons in the large N_c limit of QCD are described by non-relativistic Hartree equations

$$[\nabla^2 + 2m\epsilon + \varphi(\boldsymbol{r})]\chi = 0 \tag{15}$$

where

$$\varphi(\boldsymbol{r}) = 2mg'^2\int \mathrm{d}^3\boldsymbol{r}'\,\frac{|\chi(\boldsymbol{r}')|^2}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{16}$$

ϵ is the binding energy, m is the quark mass and χ is the non-relativistic single-quark wavefunction which is normalized according to $\int \mathrm{d}^3\boldsymbol{r}|\chi(\boldsymbol{r})|^2 = 1$. Very briefly the reason for this is the following. The colour part of the baryonic wavefunction is a singlet, which is antisymmetric, and hence the rest of the wavefunction must be symmetric. The colour and spin degrees of freedom can be neglected in the lowest order of approximation and, at the scale of hadronic bound states, the linear (confining) part of the potential can also be neglected. Neglecting the relativistic kinematics and certain many-particle effects, the resulting equations are the non-relativistic Hartree equations (15)–(16) for N identical quarks. In this paper we will include the relativistic kinematics and solve the relativistic Hartree equations (6)–(7), since our method works equally well in this case.

Thus, applied to the quark model, $V(r)$ in (14) with $\mu_0 = 0$ can be interpreted as an effective potential in which dressed constituent quarks are bound. The quark–quark potential for point-like quarks is $g^2/[4\pi(2m)^2 r]$, which is obtained by substituting $|\psi_i(r')|^2 = \delta(r')/2m$ in (14). Comparing this potential with the QCD motivated quark–quark potential for point-like quarks, $4\alpha_s/3r$, we obtain

$$\frac{g^2}{4\pi 4m^2} = \frac{4\alpha_s(m)}{3} \tag{17}$$

where $\alpha_s(m)$ is the strong running coupling constant of QCD taken at the mass m of the constituent quark. Equation (17) can be used to relate the results obtained for the present model to the large N_c limit of QCD. The relation to the non-relativistic Hartree equations (15)–(16) is determined by $g'^2 = g^2 N/4\pi 4m|E|$ and $2m\epsilon = -\gamma^2$. In [2] two main conclusions were drawn in arriving at Hartree equations (15)–(16). First, provided that g' does not depend on N_c (i.e. $g \propto 1/\sqrt{N_c}$), the baryon masses increase linearly with N_c. This is an immediate consequence of (12). Second, under the same provision, the size and shape of the baryons do not depend on N_c. This follows from (13) for the size and from (10)–(11) for the shape. However, to derive further conclusions concerning, for instance, the mass spectrum or the dependence of the mass on the coupling constant, requires the solution of the system (10)–(11). We will take up these issues here in a more general context where μ_0 is arbitrary. Besides the connection to QCD, the model is useful in its own right since it is self-consistent, the dimension is natural $(3 + 1)$ and the interaction is realistic (exchange of particles) and includes the self-interaction. Therefore, it allows us to study the internal dynamical properties of particles such as the relation between the bare mass m_0 and the physical mass m, or the dependence of m on the coupling constant g.

The main results of this paper are as follows.

(i) *Mass.* For $\mu_0 = 0$ there is a doublet of spherically symmetric solutions in the $N = 1$ sector, which corresponds to a doublet of elementary particles with masses m and m^*, where $m \leqslant m^*$. Consequently the composite particle with N constituents possesses $N + 1$ states with spherical symmetry, which can be classified according to how many of the constituents are in the excited state. The lowest state of the composite particle on the energy scale is the ground state with the mass M_N given by

$$M_N = N m_0 \frac{\sqrt{2}}{3} \left(\left\{ 1 + \left[1 - \left(\frac{g^2 N}{4\pi m_0^2 \alpha_0} \right)^2 \right]^{1/2} \right\}^{1/2} + \frac{4\pi m_0^2 \alpha_0}{g^2 N} \left\{ 1 - \left[1 - \left(\frac{g^2 N}{4\pi m_0^2 \alpha_0} \right)^2 \right]^{1/2} \right\}^{1/2} \right)$$

where α_0 is a numerical constant.

For $\mu_0 \neq 0$ there is a triplet of spherically symmetric solutions in the $N = 1$ sector, which corresponds to a triplet of elementary particles with masses m, m^* and m^{**}, where $m \leqslant m^*$ and $m \leqslant m^{**}$. Consequently the composite particle with N constituents now possesses $N(N+1)/2+N+1$ states with spherical symmetry, which again can be classified according to how many of the constituents occupy one or other excited states. The dependence of M/Nm_0 on $g^2N/4\pi m_0^2$ for various fixed values of μ_0/m_0 is illustrated in figure 1(a). In this figure the ground state is plotted with a full curve, the excited state, having all of the constituents residing in the state $*$, is plotted with a broken curve and the excited state, having all of the constituents residing in the state $**$, is plotted with a chain curve. Notice that, for each value of μ_0/m_0, the mass M of the ground state acquires a local minimum at the maximally allowed value of $g^2/4\pi m_0^2$ (dotted vertical line), which becomes the absolute minimum $M/Nm_0 = 2\sqrt{2}/3$ in the case $\mu_0 = 0$ (dotted horizontal line). The appearance of such well pronounced minima of the mass may be a phenomenon which is more general than one restricted to the present model (see [6], for instance) and hence may have some deeper significance (some speculations were discussed in [3]).

(ii) *Size.* For $\mu_0 = 0$ the size parameter of the ground state is given by

$$r_0 = \frac{\sqrt{2}\delta_0}{m_0}\frac{4\pi m_0^2}{g^2N}\left\{1+\left[1-\left(\frac{g^2N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2}$$

where δ_0 and α_0 are certain numerical constants. For the ground state $r_0 \to \infty$ as $g/m_0 \to 0$. This singularity disappears when $\mu_0 \neq 0$. For the case of minimal mass the size parameter of the ground state also becomes minimized too and is given by $r_0 = \sqrt{2}\delta_0/\alpha_0 m_0$.

(iii) *Stability.* The stability properties are investigated according to three independent criteria. The first one is based on the mass defect $\Delta M = Nm - M$, where M is the mass of the composite particle and m denotes the mass of its free constituents. We found that the ground state is stable against disintegration, i.e. $\Delta M > 0$.

If the mass of a particle is larger than its bare mass, then such a particle is unstable. The reason is that, in this case, some of its continuum states (for instance the states in (5)) are energetically preferable to the bound state of the particle. According to this second criterion, in the case $\mu_0 = 0$ the ground state is stable. For $\mu_0 \neq 0$ the ground state is unstable for smaller values of the coupling constant and stable for larger values (see figure 1(a)). For $\mu_0 > 0.36m_0$ the ground state becomes unstable for all possible values of the coupling constant.

The third stability criterion is based on $\delta^2 E_{\text{total}} > 0$, which means that a bound-state solution is locally stable if it corresponds to a local minimum of the total energy ($\delta E_{\text{total}} = 0$). This criterion was shown by Rosen [4] to be necessary and sufficient for a dynamical stability in the sense of Liapunov. In the present paper we prove that for $\mu_0 < \sqrt{2}m_0$ the ground state is locally stable. Moreover, it is proved that our model is the only option with locally stable bound states among the class of theories based on Lagrangians of the form (1), where p and q are arbitrary real positive constants.

(iv) *Existence condition.* A short-range interaction cannot produce a bound state unless the strength of the interaction is sufficiently large, i.e. the coupling constant is larger than a certain limit. If the system is relativistic then, in addition to this constraint, the coupling constant must be smaller than a certain limit. For the present model this existence condition is given by

$$\frac{1}{N}\frac{g_0^2}{4\pi m_0^2} \leqslant \frac{g^2}{4\pi m_0^2} \leqslant \frac{1}{N}\frac{g_1^2}{4\pi m_0^2}$$

where g_0 and g_1 are certain functions of μ_0/m_0 and are independent of N.

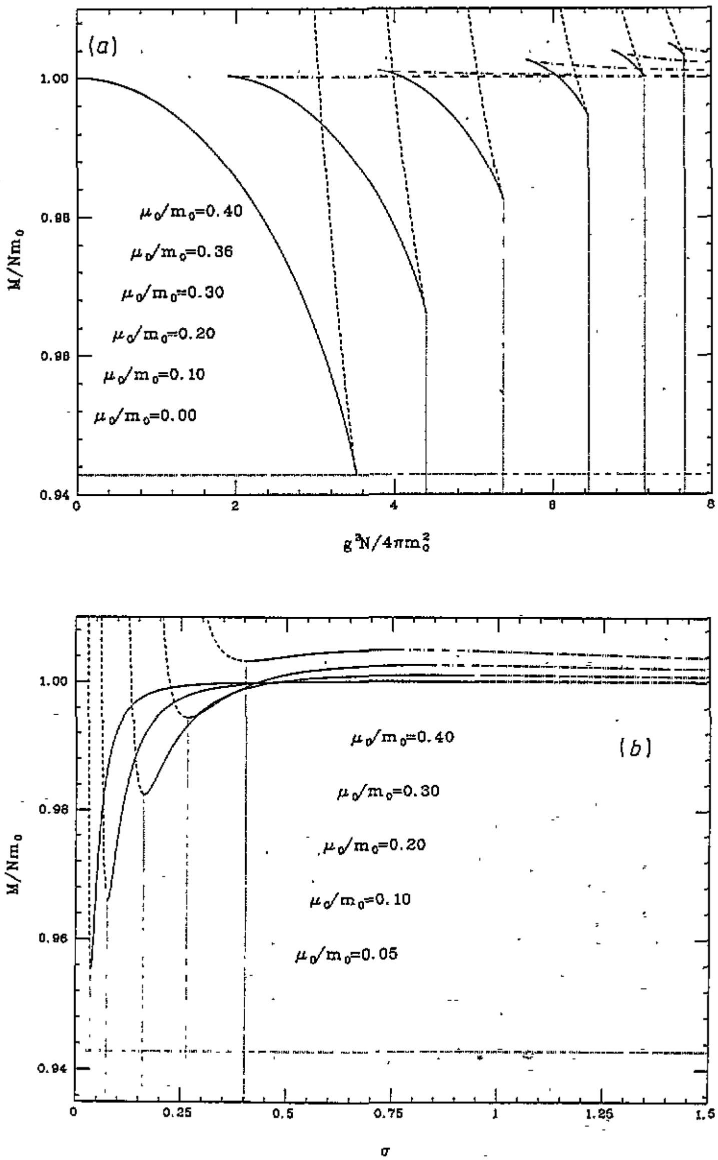

Figure 1. (a) The mass to bare mass ratios M_N/Nm_0 (full curve), M_{0NN}/Nm_0 (broken curve) and M_{N0N}/Nm_0 (chain curve) as functions of $g^2N/4\pi m_0^2$ for various fixed values of μ_0/m_0. Dotted vertical lines correspond to the maximal values of $g^2N/4\pi m_0^2$, the dotted horizontal line corresponds to the absolute minimum of the mass $M/Nm_0 = 2\sqrt{2}/3$, which occurs at $\mu = 0$. (b) The mass to bare mass ratios M/Nm_0 (full curve), M_{0NN}/Nm_0 (broken curve) and M_{N0N}/Nm_0 (chain curve) as functions of σ for various fixed values of μ_0/m_0. Dotted vertical lines correspond to the maximal values of $g^2N/4\pi m_0^2$ ($\sigma = \sigma_1$), the dotted horizontal line corresponds to the absolute minimum of the mass $M/Nm_0 = 2\sqrt{2}/3$, which occurs at $\mu = 0$.

(v) *Wavefunctions.* Equations (10) and (11) are solved using an integral transform method. The resulting wavefunctions are plotted in figures 2(a) and (b) for various values of the parameter $\sigma = \mu_0/2\gamma$, where $\gamma^2 = m_0^2 - E^2 \geqslant 0$. More precisely, the scaled functions $|E|^{1/2}\psi(r)/\gamma^{3/2}$ and $\phi(r)/\gamma^2$ are plotted, since in this form the wavefunctions are both dimensionless and depend on the dimensionless variables γr and σ only. Notice that $\phi(r) > 0$ for all r. There are no bound-state solutions for the case $\phi(r) < 0$.

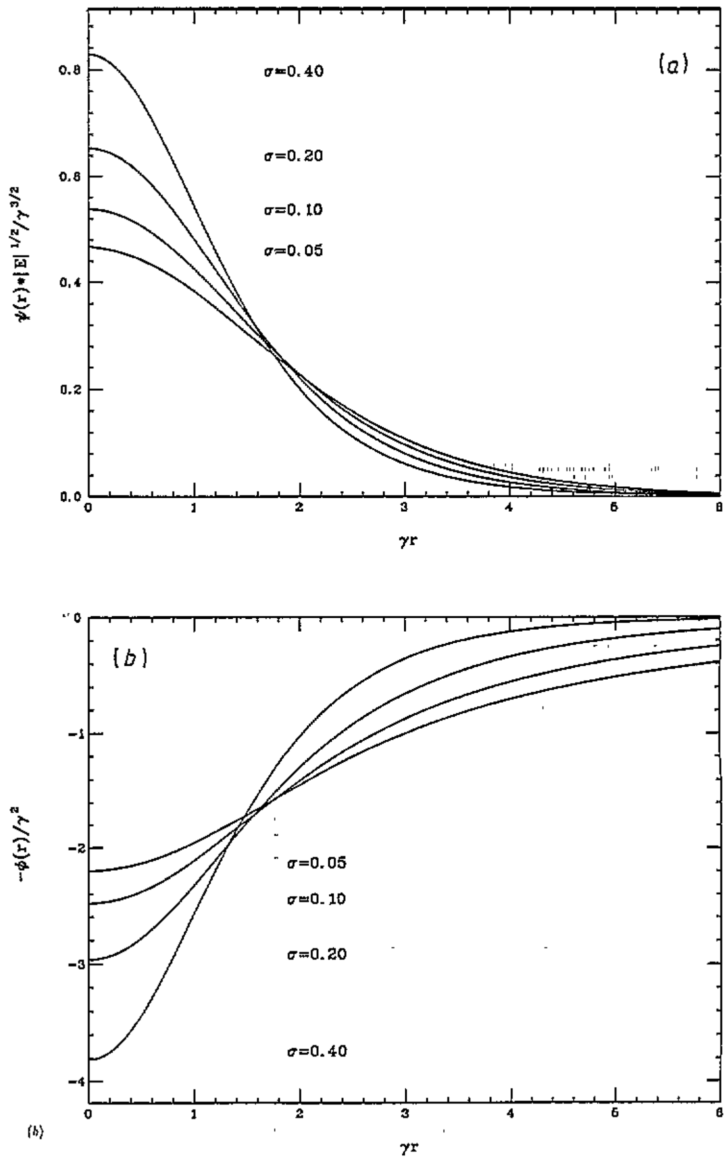

Figure 2. (a) The wavefunction $\psi(r)$ plotted as a function of the dimensionless variable γr for various values of the parameter $\sigma = \mu_0/2\gamma$, where $\gamma = (m_0^2 - E^2)^{1/2}$. The function is scaled by an appropriate factor in order to become a dimensionless wavefunction of only two variables γr and σ. The latter depends only on the fundamental parameters of the model, such as the coupling constant g. (b) The same as in (a) for the potential function $-\phi(r)$.

2. Integral transform

In [5] we discussed a method for solving the Schrödinger equation, which is most suitable if at large distances the potential is proportional to an exponential function. It follows from (11) that this is the case in the present situation and hence, according to [5] the integral transforms appropriate for (10) and (11) are

$$\psi(r) = a_0 e^{-\gamma r} \int_0^\infty d\mu\, e^{-\mu r} \varrho(\mu) \tag{18}$$

and

$$\phi(r) = b_0 e^{-\mu_0 r} \int_0^\infty d\mu\, e^{-\mu r} \mathcal{V}(\mu) \tag{19}$$

where a_0 is a normalization constant and b_0 is a constant. Substituting (18) into (10) and (19) into (11), we obtain

$$\varrho(\mu) = 1 - b_0 \theta(\mu - \mu_0) \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'(\mu' + 2\gamma)} \frac{\partial}{\partial \mu'} \int_{\mu_0}^{\mu'} d\mu''\, \mathcal{V}(\mu' - \mu'') \varrho(\mu'' - \mu_0) \tag{20}$$

and

$$\mathcal{V}(\mu) = 1 - \frac{g^2 N a_0^2}{4\pi b_0} \theta(\mu + \mu_0 - 2\gamma) \int_{2\gamma - \mu_0}^{\mu} \frac{d\mu'}{\mu'(\mu' + 2\mu_0)} \frac{\partial}{\partial \mu'}$$
$$\times \int_{2\gamma - \mu_0}^{\mu'} d\mu''\, \varrho(\mu' - \mu'') \varrho(\mu'' + \mu_0 - 2\gamma) \tag{21}$$

where $\theta(\mu - a)$ is a step function which vanishes for $\mu < a$, equals $\frac{1}{2}$ for $\mu = a$ and is equal to 1 otherwise and $a = \mu_0$ or $a = 2\gamma - \mu_0$ respectively. In deriving (21) we have assumed that $\mu_0/2\gamma \leqslant 1$. It will be shown below that this implies that we are restricting ourselves to the case $\mu_0 \leqslant 2m_0$. Now we introduce the dimensionless variables

$$s = \frac{\mu}{2\gamma} \qquad \sigma = \frac{\mu_0}{2\gamma} \qquad \eta = \frac{b_0}{2\gamma} \qquad \zeta = \frac{g^2 N a_0^2}{16\pi \gamma^2} \tag{22}$$

and redefine the functions ϱ and $\mathcal{V}$ in terms of these variables

$$R_\sigma(\eta, \zeta, s) = \varrho(\mu) \qquad \mathcal{V}_\sigma(\eta, \zeta, s) = \mathcal{V}(\mu). \tag{23}$$

Equations (20) and (21) become

$$R_\sigma(\eta, \zeta, s) = 1 - \eta \theta(s - \sigma) \int_\sigma^s \frac{dt}{t(t+1)} \frac{\partial}{\partial t} \int_\sigma^t dt'\, \mathcal{V}_\sigma(\eta, \zeta, t - t') R_\sigma(\eta, \zeta, t' - \sigma) \tag{24}$$

and

$$\mathcal{V}_\sigma(\eta, \zeta, s) = 1 - \frac{\zeta}{\eta} \theta(s + \sigma - 1) \int_{1-\sigma}^{s} \frac{dt}{t(t + 2\sigma)} \frac{\partial}{\partial t}$$
$$\times \int_{1-\sigma}^{t} dt'\, R_\sigma(\eta, \zeta, t - t') R_\sigma(\eta, \zeta, t' + \sigma - 1) \tag{25}$$

respectively. Substituting (25) in (24) and performing an integration by parts, the dependence on $\mathcal{V}_\sigma(\eta, \zeta, s)$ is eliminated and we obtain

$$R_\sigma(\eta,\zeta,s) = 1 - \eta\theta(s-\sigma)\int_\sigma^s \frac{\mathrm{d}t}{t(t+1)} R_\sigma(\eta,\zeta t-\sigma) + \zeta\theta(s-1)\int_1^s \frac{\mathrm{d}t}{t(t+1)}$$
$$\times \int_1^t \mathrm{d}t' \, \frac{R_\sigma(\eta,\zeta,t-t')}{t'^2-\sigma^2} \frac{\partial}{\partial t'} \int_1^{t'} \mathrm{d}t'' \, R_\sigma(\eta,\zeta,t'-t'') R_\sigma(\eta,\zeta,t''-1). \quad (26)$$

We return now to the question of boundary conditions for (6) and (7) or, equivalently, for (10) and (11). Since both (10) and (11) are second-order equations, they must be supplemented by two conditions each. We first consider the case of $\psi(r)$ where, as usual, the conditions are: $\psi(\infty) = 0$ and $|\psi(0)| < \infty$. The first condition has been taken into account by (18). As for the second one, in order to translate it to a suitable and practical condition, we write (18) using (22) and (23) as

$$\psi(r) = 2\gamma a_0 \mathrm{e}^{-\gamma r} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-2\gamma r s} R_\sigma(\eta,\zeta,s) \quad (27)$$

and replace $|\psi(0)| < \infty$ by

$$\left| \int_0^{s_0} \mathrm{d}s \, \mathrm{e}^{-2\gamma r_0 s} R_\sigma(\eta,\zeta,s) \right| < \Lambda \quad (28)$$

where Λ is some suitable real number. This condition becomes the true boundary condition as $s_0 \to \infty$ and then $r_0 \to 0$, but for a finite accuracy result some finite values of s_0 and r_0 are sufficient. The order of the limits cannot be interchanged since $R_\sigma(\eta, \zeta, s)$ as a function of s does not vanish at infinity (its behaviour in the large s regime can be described roughly as $s^a \cos(bs + c)$ with some constants a, b and c). The conditions for $\phi(r)$ are the same: $\phi(\infty) = 0$ and $|\phi(0)| < \infty$. Again, the first condition has been taken into account by (19), whereas for the second it follows from

$$\phi(r) = 2\gamma b_0 \mathrm{e}^{-\mu_0 r} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-2\gamma r s} \mathcal{V}_\sigma(\eta,\zeta,s) \quad (29)$$

that $|\phi(0)| < \infty$ can be replaced by

$$\left| \int_0^{s_0} \mathrm{d}s \, \mathrm{e}^{-2\gamma r_0 s} \mathcal{V}_\sigma(\eta,\zeta,s) \right| < \Lambda \quad (30)$$

with some finite values for r_0 and s_0 depending on the accuracy to be achieved. Equations (28) and (30) show that the constants η, ζ and σ are not independent.

Thus, our next task is to solve (26) with parameters η, ζ and σ satisfying (28) and (30). This will be achieved in two steps: first we solve (26) regarding η, ζ and σ as independent and then for each given σ we determine η and ζ satisfying (28) and (30). Thus, at this second step η and ζ become functions of σ.

3. Solution of integral equation

To obtain the solution of (26) we make the ansatz

$$
\begin{aligned}
R_\sigma(\eta, \zeta, s) &= \sum_{n=0}^{[s/\sigma]} \sum_{m=0}^{[s-n\sigma]} (-\eta)^n \zeta^m \varphi_{nm}(s - n\sigma - m, \sigma) \\
&\equiv \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-\eta)^n \zeta^m \theta(s - n\sigma - m) \varphi_{nm}(s - n\sigma - m, \sigma)
\end{aligned}
\tag{31}
$$

where a number in square brackets represents the largest integer which is smaller than or equal to that number. Notice that, for a fixed $s < \infty$, the right-hand side of the above equation is a finite sum, so the question of convergence does not appear. Substituting (31) into (26) and comparing equal powers of η and ζ, we obtain the recurrence relation for the functions φ_{nm}

$$
\begin{aligned}
\varphi_{nm}(y, \sigma) = \int_0^y \frac{\mathrm{d}t}{(t + n\sigma + m + \frac{1}{2})^2 - \frac{1}{4}} \Bigg(\varphi_{n-1,m}(t, \sigma) + \sum_{k=0}^{n} \sum_{l=1}^{m} \int_0^t \mathrm{d}t' \, \frac{\varphi_{n-k,m-l}(t - t', \sigma)}{(t' + k\sigma + l)^2 - \sigma^2} \\
\times \sum_{i=0}^{k} \sum_{j=1}^{l} \frac{\partial}{\partial t'} \int_0^{t'} \mathrm{d}t'' \varphi_{k-i,l-j}(t' - t'', \sigma) \varphi_{i,j-1}(t'', \sigma) \Bigg)
\end{aligned}
\tag{32}
$$

where $n = 0, 1, \ldots, m = 0, 1, \ldots$ and

$$
\varphi_{-1,m}(y, \sigma) = 0 \qquad \varphi_{00}(y, \sigma) = 1 \tag{33}
$$

which guarantees that the ansatz is, in fact, a solution. Here and in what follows we have adopted the convention that if the upper bound of a sum is less than the lower bound then the contribution of the sum is zero. Notice that for fixed n and m the evaluation of the right-hand side does not require knowledge of the left-hand side. Also notice that the ansatz has been chosen so that the recurrence relation does not depend on the parameters η and ζ. Therefore, once the φ_{nm} functions are calculated, (26) is solved for any η and ζ and we can use this solution to find the values of η and ζ satisfying the boundary conditions (28) and (30). Equation (31) combined with the above recurrence relation constitutes the exact analytical solution of (26).

The recurrence relation (32) can be iterated, so that finally no φ_{nm} functions will appear on the right-hand side. For instance,

$$
\begin{aligned}
\varphi_{10}(y, \sigma) &= \ln \frac{(\sigma + 1)(y + \sigma)}{\sigma(y + \sigma + 1)} \\
\varphi_{01}(y, \sigma) &= \frac{1}{2\sigma} \int_0^y \frac{\mathrm{d}t}{(t + 1)(t + 2)} \ln \frac{(t + 1 - \sigma)(1 + \sigma)}{(t + 1 + \sigma)(1 - \sigma)}.
\end{aligned}
\tag{34}
$$

However, for higher n and m one obtains multiple-integral representations of φ_{nm} which are not useful for our purposes. Since one cannot express the functions φ_{nm} in terms of elementary or special functions, one has to devise a numerical algorithm for their evaluation. A suitable algorithm is given in the appendix. As a by-product of this algorithm, we also

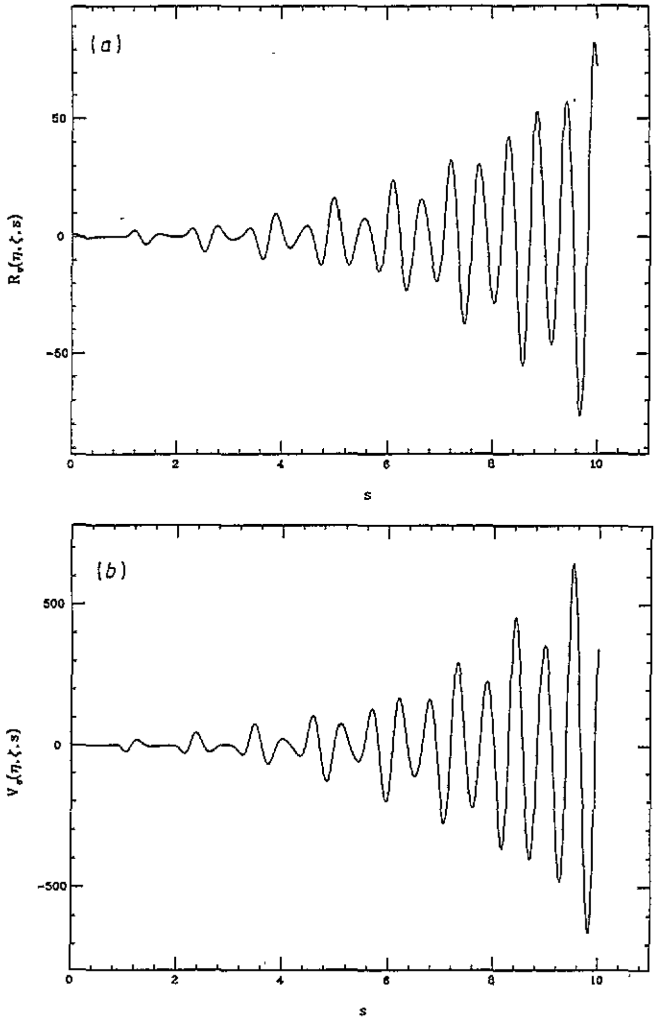

Figure 3. (*a*) The function $R_\sigma(\eta, \zeta, s)$ as a function of s for $\sigma = 0.1$ and $\eta = 2.448(5)$, $\zeta = 582.(3)$. The constants η and ζ were determined from the boundary conditions (28) and (30). (*b*) The same as in (*a*) for the function $\mathcal{V}_\sigma(\eta, \zeta, s)$.

obtain a function $f_{nm}(s, \sigma)$, which is very useful since $\mathcal{V}_\sigma(\eta, \zeta, s)$ is related to it in a simple way:

$$\mathcal{V}_\sigma(\eta, \zeta, s) = 1 - \frac{1}{\eta} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (-\eta)^n \zeta^m \theta(s + \sigma - n\sigma - m) F_{nm}(s + \sigma - n\sigma - m, \sigma) \tag{35}$$

where

$$F_{nm}(y, \sigma) = \int_0^y \mathrm{d}t\, f_{nm}(t, \sigma). \tag{36}$$

Notice that, for a fixed $s < \infty$, the right-hand side of (35) is a finite sum.

Once the functions $\varphi_{nm}(y, \sigma)$ and $F_{nm}(y, \sigma)$ are calculated we can use them to calculate the function $R_\sigma(\eta, \zeta, s)$ according to (31) and the function $\mathcal{V}_\sigma(\eta, \zeta, s)$ according to (35), and to determine the values of η and ζ for which the boundary conditions (28) and (30) are fulfilled. A typical result is illustrated in figure 3.

4. Wavefunctions and binding energies

To determine the wavefunction $\psi(r)$ of a constituent we have to determine the normalization constant a_0. Using the normalization condition (8) and equation (27), we obtain

$$a_0^2 = \frac{2\gamma\zeta}{\alpha|E|} \tag{37}$$

where

$$\alpha = 16\zeta \int_0^\infty \mathrm{d}\xi\, \xi^2 \Psi_\sigma^2(\xi) \qquad \Psi_\sigma(\xi) = \mathrm{e}^{-\xi} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-2\xi s} R_\sigma(\eta, \zeta, s). \tag{38}$$

Notice that σ is the only parameter upon which α depends. Using (38) and the function $R_\sigma(\eta, \zeta, s)$ calculated above, α can be determined. A useful fit is

$$\alpha = \alpha_0 + \alpha_1\sigma + \alpha_2\sigma^2 \qquad \sigma \leqslant 1 \tag{39}$$

where $\alpha_0 = 3.52(2)$, $\alpha_1 = 10.9(2)$ and $\alpha_2 = 3.82(9)$. Using (37) and (27) we can write the wavefunction in a dimensionless form:

$$\frac{|E|^{1/2}}{\gamma^{3/2}}\psi(r) = 2\sqrt{\frac{2\zeta}{\alpha}}\mathrm{e}^{-\gamma r} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-2\gamma r s} R_\sigma(\eta, \zeta, s). \tag{40}$$

Notice that the right-hand side of (40) is a function of the dimensionless quantities γr and σ only. Similarly, using (29) and (22) the potential function $\phi(r)$ can also be written in the dimensionless form

$$\frac{1}{\gamma^2}\phi(r) = 4\eta \mathrm{e}^{-2\sigma\gamma r} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-2\gamma r s} V_\sigma(\eta, \zeta, s). \tag{41}$$

In figures 2(*a*) and (*b*) we plotted the wavefunction and the potential function according to (40) and (41) respectively for various values of σ.

We define the binding energy $E - m_0$ of a constituent particle as the difference between its constituent energy E and its bare mass m_0. To determine the constituent energy E we substitute (37) into the expression for ζ provided by (22) and using $\gamma^2 = m_0^2 - E^2$ obtain

$$|E|(m_0^2 - E^2)^{1/2} = \frac{1}{2}\frac{g^2 N}{4\pi\alpha}. \tag{42}$$

Solving for $|E|$ we obtain two solutions:

$$\varepsilon^* = \frac{m_0}{\sqrt{2}} \left\{ 1 - \left[1 - \left(\frac{g^2 N}{4\pi m_0^2 \alpha} \right)^2 \right]^{1/2} \right\}^{1/2} \tag{43}$$

and

$$\varepsilon = \frac{m_0}{\sqrt{2}} \left\{ 1 + \left[1 - \left(\frac{g^2 N}{4\pi m_0^2 \alpha} \right)^2 \right]^{1/2} \right\}^{1/2} \tag{44}$$

which are real if $g^2N/4\pi m_0^2\alpha \leqslant 1$. The first root is the constituent energy of each constituent in the excited state, while the second root corresponds partly to the ground state and partly to the excited state, depending on the value of σ. Equations (43) and (44) imply that.

$$0 < \frac{\varepsilon^*}{m_0} \leqslant \frac{1}{\sqrt{2}} \tag{45}$$

and

$$\frac{1}{\sqrt{2}} \leqslant \frac{\varepsilon}{m_0} < 1. \tag{46}$$

Notice that ε^* is not larger than ε, and nevertheless ε^* is assigned entirely to the excited state. The reason is that, as will be shown below, the total energy (the mass) corresponding to the bound state with constituent energy ε^* is always larger than in the case of a bound state with constituent energy ε.

In the case when $\mu_0 = 0$, the function $\alpha = \alpha_0 = 3.52(2)$ is a pure numerical constant and hence (43) and (44) are explicit formulae. Otherwise, α is a function of σ which itself is a function of both parameters μ_0/m_0 and g. We will not need the explicit form of this function, since σ will serve only as a parameterization variable at fixed μ_0/m_0. The allowed range of σ follows from

$$\sigma = \frac{1}{2}\frac{\mu_0}{m_0}\frac{1}{[1-(E/m_0)^2]^{1/2}} \tag{47}$$

which is obtained from the definition $\sigma = \mu_0/2\gamma$ and from $\gamma^2 = m_0^2 - E^2$. Substituting (45) and (46) in (47), we obtain respectively

$$\frac{1}{2}\frac{\mu_0}{m_0} < \sigma \leqslant \frac{1}{\sqrt{2}}\frac{\mu_0}{m_0} \tag{48}$$

and

$$\frac{1}{\sqrt{2}}\frac{\mu_0}{m_0} \leqslant \sigma < \infty. \tag{49}$$

The coupling constant $g^2N/4\pi m_0^2$, for instance, is parameterized by σ at fixed μ_0/m_0 via

$$\frac{g^2N}{4\pi m_0^2} = \frac{\alpha\mu_0}{\sigma m_0}\left[1 - \frac{1}{4}\left(\frac{\mu_0}{\sigma m_0}\right)^2\right]^{1/2}. \tag{50}$$

Equation (50) is obtained if one combines (42) and (47) to eliminate E. Using (50) we can convert the dependence on σ in (43) and (44) to a dependence on $g^2N/4\pi m_0^2$ and μ_0/m_0. The result, ε/m_0 and ε^*/m_0 as functions of $g^2N/4\pi m_0^2$ and μ_0/m_0, is plotted in figure 4. The dotted line indicates the value $1/\sqrt{2}$ which is the boundary value deviding the two functions.

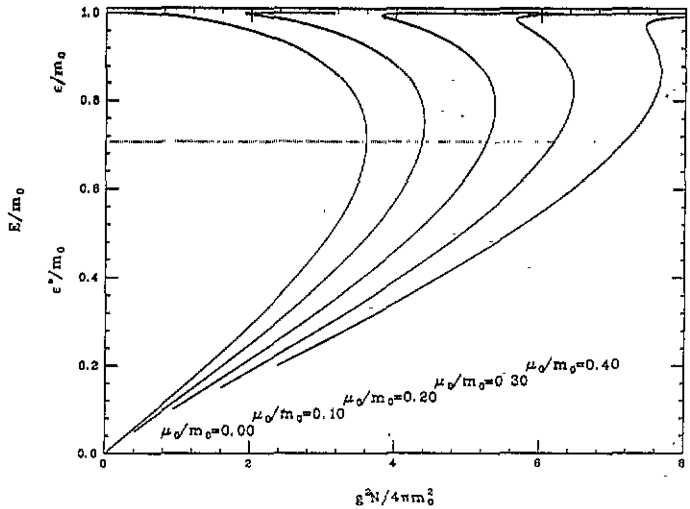

Figure 4. The constituent energies ε/m_0 and ε^*/m_0 as functions of the coupling constant $g^2N/4\pi m_0^2$ at various fixed values of μ_0/m_0. The dotted line indicates the value $1/\sqrt{2}$ which is the boundary value dividing the two functions.

5. Existence condition

Equations (43) and (44) imply that

$$0 < \frac{g^2}{4\pi m_0^2} \leqslant \frac{\alpha}{N} \tag{51}$$

since otherwise the constituent energy becomes a complex number, in which case the bound state does not exist. In the case when $\mu_0 = 0$, $\alpha = \alpha_0 = 3.52(2)$ is a pure numerical constant and hence (51) is an explicit necessary and sufficient condition for the existence of bound states.

In the case when $\mu_0 \neq 0$ (51) is not suitable as an explicit existence condition, since in this case α is not a pure number but rather a function of σ and thus of $g^2N/4\pi m_0^2$ and μ_0/m_0. To obtain the existence condition for the case when $\mu_0 \neq 0$, we write (50) as

$$\frac{g^2}{4\pi m_0^2} = \frac{\alpha}{N}\frac{\mu_0}{\sigma m_0}\left[1 - \frac{1}{4}\left(\frac{\mu_0}{\sigma m_0}\right)^2\right]^{1/2} \leqslant \frac{\alpha}{N} \tag{52}$$

which is the parametrization of the coupling constant in terms of σ at fixed μ_0/m_0 and N. The inequality on the right-hand side of (52) becomes an equality for $\sigma = \mu_0/\sqrt{2}m_0$. Since σ varies in a definite range given in (48) and (49), equation (52) determines the allowed range of the coupling constant, in which a bound-state solution is possible. In figure 5 we plot $g^2N/4\pi m_0^2$ as a function of σ for a few typical values of μ_0/m_0 (full curves). Notice that $g^2N/4\pi m_0^2$ possesses a local maximum in the region of smaller σ and a local minimum in the region of larger σ. From (52) and (39) we obtain

$$\sigma_1 = \frac{1}{\sqrt{2}}\frac{\mu_0}{m_0} + \frac{\alpha_1}{8\alpha_0}\left(\frac{\mu_0}{m_0}\right)^2 + \mathrm{O}\left(\left(\frac{\mu_0}{m_0}\right)^2\right) \tag{53}$$

for the location $\sigma = \sigma_1$ of the local maximum and

$$\sigma_0 = \sqrt{\frac{\alpha_0}{\alpha_2}} + \mathrm{O}\left(\left(\frac{\mu_0}{m_0}\right)^2\right) \tag{54}$$

for the location $\sigma = \sigma_0$ of the local minimum. The respective accuracies of (53) and (54) are sufficient for all practical purposes. This can be judged by looking at figure 5, where σ_1 from (53) is plotted as a vertical dotted line and $\sigma_0 = \sqrt{\alpha_0/\alpha_2}$ is plotted as a vertical broken line. The maximum and the minimum split the allowed range of $g^2/4\pi m_0$ into three regions: one to the left of the maximal value

$$0 < \frac{g^2}{4\pi m_0^2} \leqslant \frac{1}{N}\frac{g_1^2}{4\pi m_0^2} \qquad \frac{1}{2}\frac{\mu_0}{m_0} < \sigma \leqslant \sigma_1 \tag{55}$$

one between the maximal value and the minimal value

$$\frac{1}{N}\frac{g_0^2}{4\pi m_0^2} \leqslant \frac{g^2}{4\pi m_0^2} \leqslant \frac{1}{N}\frac{g_1^2}{4\pi m_0^2} \qquad \sigma_1 \leqslant \sigma \leqslant \sigma_0 \tag{56}$$

and one to the right of the minimal value

$$\frac{1}{N}\frac{g_0^2}{4\pi m_0^2} \leqslant \frac{g^2}{4\pi m_0^2} \qquad \sigma_0 \leqslant \sigma \tag{57}$$

where

$$\frac{g_1^2}{4\pi m_0^2} = \alpha(\sigma_1)\frac{\mu_0}{\sigma_1 m_0}\sqrt{1 - \frac{1}{4}\left(\frac{\mu_0}{\sigma_1 m_0}\right)^2} = \alpha_0 + \frac{\alpha_1}{\sqrt{2}}\frac{\mu_0}{m_0} + \mathrm{O}\left(\left(\frac{\mu_0}{m_0}\right)^2\right) \tag{58}$$

$$\frac{g_0^2}{4\pi m_0^2} = \alpha(\sigma_0)\frac{\mu_0}{\sigma_0 m_0}\sqrt{1 - \frac{1}{4}\left(\frac{\mu_0}{\sigma_0 m_0}\right)^2} = \left(2\sqrt{\alpha_0\alpha_2} + \alpha_1\right)\frac{\mu_0}{m_0} + \mathrm{O}\left(\left(\frac{\mu_0}{m_0}\right)^3\right) \tag{59}$$

and α is given in (39). These three branches of the coupling constant correspond to three different states (particles), and below it will be shown that (56) corresponds to the ground state while (55) and (57) each correspond to a different excited state. Hence for the case of $\mu_0 \neq 0$ the model predicts a triplet of particles. Notice that in the case when $\mu_0 = 0$ (56) becomes

$$0 < \frac{g^2}{4\pi m_0^2} \leqslant \frac{\alpha_0}{N}. \tag{60}$$

If for certain μ_0/m_0 and N it happens that $g^2/4\pi m_0^2$ is outside of the bounds defined in (56) then a ground-state solution and a particle associated with it do not exist and there are no stable particles in that case. Thus (56) is an explicit condition for the existence of a ground state and its associated particle. Notice that (56) is in qualitative agreement with the non-relativistic case of the Yukawa potential [5], where the ground-state solution is possible only if the coupling constant $f^2/4\pi$ is larger than $(1.6798/2)(\mu_0/m_0)$.

In the case when the condition (56) for the existence of the ground state is violated while (55) is fulfilled, the particle associated with the ground state does not exist while the excited state does. However, as will be shown below, the mass of such a state is larger than its bare mass. Hence this state is unstable and represents an unstable particle.

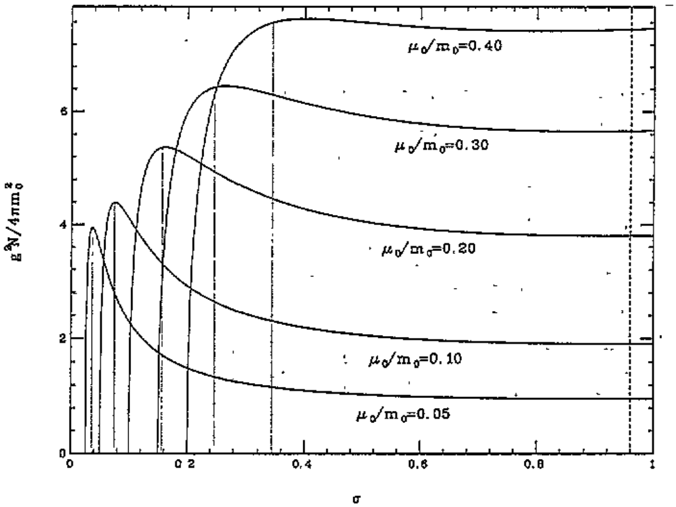

Figure 5. $g^2N/4\pi m_0^2$ as a function of σ for various values of μ_0/m_0 (full curves). The vertical dotted lines and the vertical broken line give the values of σ_1 and σ_0 for the maximum and the minimum of g respectively, according to the approximate formulae given in the text.

6. Mass and bare mass

Instead of calculating the last term in (12) directly, we can use the virial theorem (this theorem is proved in section 8) part of which is the relation

$$\frac{1}{2}\int d^3r\,\phi(r)|\psi(r)|^2 = \frac{m_0}{3}\left(\frac{m_0}{|E|} - \frac{|E|}{m_0}\right) + \frac{1}{3}\frac{\mu_0^2}{g^2N}\int d^3r\,\phi^2(r). \tag{61}$$

Substituting (61) in (12) we obtain

$$M = Nm_0\left(\frac{2}{3}\frac{|E|}{m_0} + \frac{1}{3}\frac{m_0}{|E|} + \frac{1}{3}\frac{\mu_0^2}{m_0 g^2 N}\int d^3r\,\phi^2(r)\right) > 0. \tag{62}$$

Notice that the total energy (62) is positive definite.

We first consider the case when $\mu_0 = 0$. In this case the last term in (62) vanishes and we obtain

$$M = Nm_0\frac{2}{3}\left(\frac{|E|}{m_0} + \frac{1}{2}\frac{m_0}{|E|}\right). \tag{63}$$

Substitution of (44) and (43) in (63) yields

$$M_N = Nm_0\frac{\sqrt{2}}{3}\left(\left\{1+\left[1-\left(\frac{g^2N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2} + \frac{4\pi m_0^2\alpha_0}{g^2N}\left\{1-\left[1-\left(\frac{g^2N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2}\right) \tag{64}$$

for the ground state and

$$M_{NN} = Nm_0\frac{\sqrt{2}}{3}\left(\left\{1-\left[1-\left(\frac{g^2N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2}\right.$$
$$\left.+\frac{4\pi m_0^2\alpha_0}{g^2N}\left\{1+\left[1-\left(\frac{g^2N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2}\right) \tag{65}$$

for the excited state. The masses of the corresponding constituent particles are

$$m = M_1 \qquad m^* = M_{11}. \tag{66}$$

Thus, for the case of $\mu_0 = 0$ the model predicts a doublet in the elementary particle sector ($N = 1$) with masses m, m^*. Notice that (64)–(66) imply that for fixed parameters $g^2N/4\pi m_0^2$ and Nm_0

$$M_N \leqslant M_{NN} \qquad m \leqslant m^* \tag{67}$$

despite the fact that according to (44) and (43) $\varepsilon^* \leqslant \varepsilon$. The reason is that, besides the contribution from the constituent energies (first term in (9)), the mass also receives a contribution from the energy (second term in (9)) which must be spent to create the potential in which the constituents are bound. This contribution compensates for the difference between ε^* and ε and places M_{NN} above M_N. Also notice that while $M_{NN} \to \infty$ and $m^* \to \infty$ as $g \to 0$, M_N and m stay finite.

The state with mass M_{NN} is the highest excited state with spherical symmetry, since we have assumed that all of the constituents occupy the same state and that there are only two bound states which each of the constituents can occupy. Our method, however, can be extended easily to a situation where one or more constituents occupy the excited state with the corresponding constituent energy ε^* while others remain in the other bound state. The resulting excited states will possess masses between M_N and M_{NN}:

$$M_N \leqslant M_{1N}, M_{2N}, \cdots \leqslant M_{NN}. \tag{68}$$

These are all the possible spherically symmetric states, since otherwise there would be more states than just a doublet in the $N = 1$ sector. Hence for the case of $\mu_0 = 0$, the total number of states with spherical symmetry is

$$N_{\text{states}} = N + 1. \tag{69}$$

A further consequence of (64)–(66) is

$$\Delta M_N = Nm - M_N > 0$$

which means that the ground state of the composite particle is stable against disintegration. Similarly, one obtains

$$\Delta M_{NN} = Nm^* - M_{NN} > 0 \tag{70}$$

which means that the excited state cannot decay by disintegration with emission of particles of mass m^*.

Above we discussed the stability of the composite particle from the point of view of disintegration. We now discuss the stability from the point of view of a comparison between the bare mass and the physical mass. This criterion allows us to discuss the stability of even the elementary particles of the theory. From (64)–(66) it follows that

$$0 < \frac{g^2}{4\pi m_0^2} \leqslant \frac{\alpha_0}{N} \Leftrightarrow M_N < N m_0 \tag{71}$$

for the ground state and

$$0 < \frac{g^2}{4\pi m_0^2} \leqslant \frac{\sqrt{3}}{2}\frac{\alpha_0}{N} \Leftrightarrow M_{NN} \geqslant N m_0 \tag{72}$$

$$\frac{\sqrt{3}}{2}\frac{\alpha_0}{N} < \frac{g^2}{4\pi m_0^2} \leqslant \frac{\alpha_0}{N} \Leftrightarrow M_{NN} < N m_0 \tag{73}$$

for the excited state. If we want to associate a particle with a self-interaction bound state, then the result that the physical mass is larger than the bare mass means that such a particle is unstable. The reason is that in this case some of its continuum states are energetically preferable to the bound state. Thus from (71) we conclude that the ground state M_N is always stable if it exists, while the excited state M_{NN} is unstable for smaller values of the coupling constant g in the case of (72), and stable for larger values of the coupling constant in the case of (73). The phenomenon of instability due to the mass being larger than the bare mass is not new. It was observed, for example, for the ground state in the model of [6], which is based on the nonlinear Schrödinger equation in three dimensions. In that model one can give a formal proof (see [7] for the details) that the ground state is unstable in the sense that its wavefunction suffers a collapse, i.e. being initially smooth it develops a singularity within a finite period of time when subjected to a small perturbation. This means that such a state cannot be associated with a stable free particle.

In the case of (73) the composite particle can exist as a stable (with respect to the continuum states) particle with the mass M_N or M_{NN}. For the mass difference we have

$$0 \leqslant M_{NN} - M_N < \frac{3\sqrt{2}-4}{4} M_N \tag{74}$$

which amounts to less than 6%.

We now discuss the case when $\mu_0 \neq 0$. Using (19), (22), (42) and (47), we obtain

$$\frac{1}{2}\frac{\mu_0^2}{m_0 g^2 N}\int \mathrm{d}^3 r \phi^2(r) = \left(\frac{m_0}{|E|} - \frac{|E|}{m_0}\right)\beta \tag{75}$$

where

$$\beta = \frac{16\sigma^2\eta^2}{\alpha}\int_0^\infty \mathrm{d}\xi\, \xi^2\phi_\sigma^2(\xi) \qquad \phi_\sigma(\xi) = \mathrm{e}^{-2\sigma\xi}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-2\xi s}\mathcal{V}_\sigma(\eta,\zeta,s) \tag{76}$$

and $\mathcal{V}_\sigma(\eta,\zeta,s)$ is defined in (25). Notice that β is a function of σ only. To get some idea of the behaviour of β the following fit is useful

$$\beta = \beta_0\sigma\frac{1+\beta_1\sigma}{1+\beta_2\sigma} \qquad \sigma \leqslant 1 \tag{77}$$

where $\beta_0 = 0.834$, $\beta_1 = 0.972$ and $\beta_2 = 2.93$. The accuracy of (77) is about 1%. Using (75) we can rewrite (62) as

$$M = Nm_0\frac{2}{3}\left((1-\beta)\frac{|E|}{m_0} + \frac{1}{2}(1+2\beta)\frac{m_0}{|E|}\right). \tag{78}$$

Substituting (43) and (44) in (78) we can obtain similar mass formulae to (64)–(66). More convenient formulae are obtained if, using (52), we eliminate $g^2N/4\pi m_0^2$ from (43) and (44) and substitute the result in (78). The resulting mass formulae are

$$M_N = Nm_0\Xi\left(\sigma, \frac{\mu_0}{m_0}\right) \qquad \sigma_0 \geqslant \sigma \geqslant \sigma_1 = \frac{1}{\sqrt{2}}\frac{\mu_0}{m_0} + \frac{\alpha_1}{8\alpha_0}\left(\frac{\mu_0}{m_0}\right)^2 + \mathrm{O}\left(\left(\frac{\mu_0}{m_0}\right)^2\right) \tag{79}$$

$$M_{0NN} = Nm_0\Xi\left(\sigma, \frac{\mu_0}{m_0}\right) \qquad \sigma_1 \geqslant \sigma > \frac{1}{2}\frac{\mu_0}{m_0} \tag{80}$$

and

$$M_{N0N} = Nm_0\Xi\left(\sigma, \frac{\mu_0}{m_0}\right) \qquad \infty > \sigma > \sigma_0 = \sqrt{\frac{\alpha_0}{\alpha_2}} + \mathrm{O}\left(\left(\frac{\mu_0}{m_0}\right)^2\right) \tag{81}$$

where

$$\Xi\left(\sigma, \frac{\mu_0}{m_0}\right) = \frac{1-\frac{1}{6}(1-\beta)(\mu_0/\sigma m_0)^2}{1-\frac{1}{4}(\mu_0/\sigma m_0)^2}\left[1-\frac{1}{4}\left(\frac{\mu_0}{\sigma m_0}\right)^2\right]^{1/2}$$
$$\equiv \left(1-\frac{1-\beta}{6}\left(\frac{\mu_0}{\sigma m_0}\right)^2\right)\frac{\mu_0}{\sigma m_0}\frac{4\pi m_0^2\alpha}{g^2N}. \tag{82}$$

The advantage of (79)–(81) is that, for a fixed μ_0/m_0 and Nm_0, they depend solely on σ. The dependence of M_N, M_{0NN} and M_{N0N} on σ for various fixed values of μ_0/m_0 is illustrated in figure 1(*b*). In this figure M_N/Nm_0 is plotted with a full curve, M_{0NN}/Nm_0 with the broken curve and M_{N0N}/Nm_0 with the chain curve. The dependence on σ can be converted to the dependence on $g^2N/4\pi m_0^2$ via (52)

$$\frac{g^2N}{4\pi m_0^2} = \alpha\frac{\mu_0}{\sigma m_0}\left[1-\frac{1}{4}\left(\frac{\mu_0}{\sigma m_0}\right)^2\right]^{1/2}. \tag{83}$$

The result is illustrated in figure 1(*a*). In both figures the dotted vertical lines correspond to the maximal values of $g^2N/4\pi m_0^2$ according to (52) ($\sigma = \sigma_1$), while the dotted horizontal line corresponds to the absolute minimum of the mass $M/Nm_0 = 2\sqrt{2}/3$, which occurs at $\mu_0 = 0$. Hence we observe that the mass M_N of the ground state acquires a minimum value at the maximal value of $g^2/4\pi m_0^2$. We make a similar observation with respect to the maximal value of the mass M_N: this occurs at the minimal value of $g^2/4\pi m_0^2$ according to (52) ($\sigma = \sigma_0$). Notice that, except for the case when $\mu_0/m_0 = 0$, part of M_N including the maximum resides within the continuum, i.e. above Nm_0, signalling an unstable state. This part becomes larger with increasing μ_0/m_0 and starting with $\mu_0/m_0 = 0.36$ all of the mass M_N resides within the continuum. This means that for a stable ground state the mass of the exchange particle cannot be too large: $\mu_0 < 0.36m_0$.

In the case of elementary particles of the model ($N = 1$) we have a triplet with the corresponding masses

$$m = M_1 \qquad m^* = M_{011} \qquad m^{**} = M_{101} \tag{84}$$

which are obtained by putting $N = 1$ in (79)–(81). This fact is in contrast to the $\mu_0 = 0$ case where there was a doublet, and is a clear indication that the mass is not an analytic function of μ_0 at $\mu_0 = 0$. The mass is also non-analytic at $g = g_1$, where it has a minimum, and at $g = g_0$ where it has a local maximum (see equations (58) and (59)). Otherwise it is an analytic function of $g^2N/4\pi m_0^2$ and μ_0/m_0.

The appearance of a triplet with different masses and different constituent energies at the same values of $g^2N/4\pi m_0^2$ and μ_0/m_0 indicates that there are other states with spherical symmetry, which can be labelled by the number of constituents in each of the two excited states. The lowest-mass state M_N is the ground state, where none of the constituents resides in an excited state. Then there are states $M_{k,n-k,N}$ where n constituents reside in an excited state, k of them in one of the excited states and $n - k$ in the other one. Since all these states are spherically symmetric, our method can be easily extended to include them. Basically, the only significant change in this case is the tripling of the basic equations, so that we would have six coupled equations instead of two. Since in the present case there are no other states beyond the triplet, it is fairly clear that these are all spherically symmetric states of the model and that their total number is

$$N_{\text{states}} = \tfrac{1}{2}N(N+1) + N + 1. \tag{85}$$

Thus in this model there are three elementary particles (triplet), six composite particles with two constituents (sextet), ten composite particles with three constituents (decuplet) etc.

One of the most basic properties of the masses as functions of $g^2N/4\pi m_0^2$ at fixed μ_0/m_0 is

$$\frac{g'^2N}{4\pi m_0^2} \leqslant \frac{g^2N}{4\pi m_0^2} \Rightarrow \frac{M'}{Nm_0} \geqslant \frac{M}{Nm_0} \tag{86}$$

where M' and M are any of the masses M_N, M_{N0N}, M_{0NN}, m, m^* and m^{**} evaluated at $g'^2N/4\pi m_0^2$ and $g^2N/4\pi m_0^2$ respectively. An immediate implication of this result is that

$$\Delta M_N = Nm - M_N > 0 \tag{87}$$

$$\Delta M_{0NN} = Nm^* - M_{0NN} > 0 \tag{88}$$

$$\Delta M_{N0N} = Nm^{**} - M_{N0N} > 0. \tag{89}$$

Thus none of the composite particles can decay by disintegration and, in particular, the ground state is absolutely stable whenever its mass is below the bare mass. In the case when any of the masses is larger than the corresponding bare mass, for instance $M_{N0N} \geqslant Nm_0$ always, the particle with that mass is unstable as discussed above.

One of the striking features of the mass (total energy) as a function of the coupling constant $g^2/4\pi m_0^2$ is the appearance of a strongly pronounced unique minimum at each fixed value of μ_0/m_0 and N (see figure 1(a)). These minimal values of the mass and the corresponding coupling constants can be determined from (79), (52) and (53) and, to first order in μ_0/m_0, are given by

$$M_N = Nm_0\frac{2\sqrt{2}}{3}\left(1+\frac{\beta_0}{2\sqrt{2}}\frac{\mu_0}{m_0}\right) \qquad \frac{g^2}{4\pi m_0^2} = \frac{1}{N}\left(\alpha_0+\frac{\alpha_1}{\sqrt{2}}\frac{\mu_0}{m_0}\right) \qquad N=1,2,\ldots \tag{90}$$

where $\alpha_0 = 3.52(2)$, $\alpha_1 = 10.9(2)$, $\beta_0 = 0.834$. The fact that the physical mass of a particle possesses such an absolute minimum was noticed earlier in [6] in connection with some other model, where it was emphasized that this phenomenon might be a general one and might have some deep significance (some speculations are discussed in [3]).

7. Size parameter

The size parameter r_0 has been defined in (13). Substituting (27) and (37) in (13), we obtain

$$r_0 = \frac{\delta}{\gamma} \equiv \frac{\delta}{\sqrt{m_0^2 - E^2}} \tag{91}$$

where

$$\delta = \frac{16\zeta}{\alpha}\int_0^\infty \mathrm{d}\xi\, \xi^3 \Psi_\sigma^2(\xi) \tag{92}$$

and $\Psi_\sigma(\xi)$ is defined in (38). As in the case of α and β, the function δ can also be easily computed, once the integral equation (26) above is solved. A useful fit, which is accurate to about 1%, is

$$\delta = \frac{\delta_0}{\alpha}\sqrt{1+\delta_1\sigma+\delta_2\sigma^2} \qquad \sigma \leqslant 1 \tag{93}$$

where $\delta_0 = 8.4$, $\delta_1 = 1.35$, $\delta_2 = 5.75$ and α is given in (39). Thus

$$1.3 < \delta < 2.4. \tag{94}$$

In the case when $\mu_0 = 0$, we have $\sigma = 0$ and hence using (43) and (44), we obtain

$$r_0 = \frac{\delta}{\varepsilon\varepsilon^*}\varepsilon = \frac{\sqrt{2}\delta_0}{m_0}\frac{4\pi m_0^2}{g^2 N}\left\{1+\left[1-\left(\frac{g^2 N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2} \tag{95}$$

for the size of the ground state and

$$r_0^* = \frac{\delta}{\varepsilon\varepsilon^*}\varepsilon^* = \frac{\sqrt{2}\delta_0}{m_0}\frac{4\pi m_0^2}{g^2 N}\left\{1-\left[1-\left(\frac{g^2 N}{4\pi m_0^2\alpha_0}\right)^2\right]^{1/2}\right\}^{1/2} \tag{96}$$

for the size of the excited state. Notice that $r_0 \to \infty$ as $g/m_0 \to 0$ while r_0^* stays finite. This singularity disappears when $\mu_0 \neq 0$, since in this case $g \geqslant g_0 > 0$ (see equation (59)). For the case of minimal masses, equation (90) for $\mu_0 = 0$, the sizes are

$$r_0 = r_0^* = \frac{1}{m_0}\frac{\sqrt{2}\delta_0}{\alpha_0}. \tag{97}$$

For the ground state (97) corresponds to the minimal size, while for the excited state it corresponds to the maximal size.

8. Derrick's theorem and proof of local stability

Derrick's theorem [8] refers to time-independent solutions of a class of nonlinear equations for real scalar fields and it consists of two parts. The first part is the virial theorem and the second part is the proof of local instability. We now recall Derrick's theorem and discuss some of the problems connected with it. We then generalize this theorem to a class of theories which contains the model investigated in this paper, and derive the conditions of local stability. In particular we prove the local stability of the ground-state solution studied in this paper.

Consider the Lagrangian for a real scalar field θ

$$\mathcal{L} = \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial t}\right)^2 - (\nabla\theta)^2 - f(\theta)\right) \tag{98}$$

and the corresponding equation of motion

$$\Delta\theta - \frac{\partial^2\theta}{\partial t^2} = \frac{1}{2}f'(\theta) \tag{99}$$

where f is a smooth function. A time-independent solution $\theta(\boldsymbol{r})$ of (99) corresponds to the extremum $\delta H = 0$ of the energy functional

$$H = \int \mathrm{d}^3\boldsymbol{r}[(\nabla\theta)^2 + f(\theta)] \equiv I_1 + I_2. \tag{100}$$

Using this fact and a particular form of the variation δH, Derrick proved that the kinetic part I_1 and the potential part I_2 are related according to

$$I_1 + 3I_2 = 0. \tag{101}$$

Equation (101) constitutes the virial theorem. In the case $f(\theta) \geqslant 0$ this theorem precludes the existence of time-independent solutions of (99) since in this case both $I_2 > 0$ and $I_1 > 0$, which contradicts (101). If $f(\theta) \geqslant 0$ is not valid, the energy H is not bounded from below and hence a time-independent solution of (99) can be stable at most locally. However, using a particular form of the variation, Derrick showed that

$$\delta^2 H = -2I_1 < 0. \tag{102}$$

Local stability requires $\delta^2 H > 0$ (local minimum of the total energy) for all possible variations, but to prove the local instability it is sufficient to show that $\delta^2 H \leqslant 0$ for a particular variation, so that (102) implies that all time-independent solutions of (99) are locally unstable. Equation (102) constitutes the second part of Derrick's theorem. Shortly after Derrick's paper Rosen [4] proved that $\delta^2 H > 0$ is the necessary and sufficient condition for dynamical stability in the sense of Liapunov.

In the subsequent repetitions of Derrick's theorem (see for instance [9]) the second part of Derrick's theorem was dropped and the local stability condition $\delta^2 H > 0$ replaced by the much stronger condition $f(\theta) \geqslant 0$. However, if one admits time-dependent solutions, in particular stationary bound-state solutions of the form $\Psi(\boldsymbol{r}, t) = \mathrm{e}^{-\mathrm{i}Et}\psi(\boldsymbol{r})$, then the latter condition cannot be justified in general and the former, contrary to (102), can now be proved for particular cases. Thus in particular cases, at least locally, stable bound-state solutions are possible. A simple illustration of this fact is provided by

$$(\Box - m_0^2 + gq(\Psi^*\Psi)^{q-1})\Psi = 0 \qquad 1 < q < \tfrac{5}{3} \tag{103}$$

where $\Box \equiv \Delta - \partial^2/\partial t^2$. The local stability of stationary bound states of (103) was proved in [10]. Notice that the energy corresponding to (103) is

$$H = \int \mathrm{d}^3 r \left[\nabla\Psi^* \cdot \nabla\Psi + \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t} + m_0^2\Psi^*\Psi - g(\Psi^*\Psi)^q\right] \tag{104}$$

and that the interaction potential (last term in (104)) is strictly negative (g is a positive constant) and unbounded, and nevertheless the ground state has a finite energy and is locally stable. The proof of local stability for the non-relativistic analogue of (103) and other examples of stable theories with a non-positive interaction potential can be found in [7] and references therein.

Consider now a system of N complex scalar fields $\Psi_j(r,t)$, $j = 1,\ldots,N$ and a real scalar field $\Phi(r,t)$ with the Lagrangian

$$\mathcal{L} = -\sum_{j=1}^{N}(\partial^\nu\Psi_j^*\partial_\nu\Psi_j + m_0^2\Psi_j^*\Psi_j - g\Phi^p(\Psi_j^*\Psi_j)^q) - \tfrac{1}{2}(\partial^\nu\Phi\partial_\nu\Phi + \mu_0^2\Phi^2) \tag{105}$$

and the equations of motion

$$(\Box - m_0^2 + gq\Phi^p(\Psi_j^*\Psi_j)^{q-1})\Psi_j = 0 \tag{106}$$

$$(\Box - \mu_0^2)\Phi = -gp\Phi^{p-1}\sum_{i=1}^{N}(\Psi_i^*\Psi_i)^q \tag{107}$$

where m_0, g, p and q are real positive constants and the fields Ψ_j are normalized according to

$$\int \mathrm{d}^3 r \left|\frac{\partial\Psi_j^*}{\partial t}\Psi_j - \Psi_j^*\frac{\partial\Psi_j}{\partial t}\right| = 1 \qquad j = 1,\ldots,N. \tag{108}$$

Notice that the left-hand side of (108) is a constant of motion. There are four important special cases to notice: $p = q = 1$ gives the model investigated in this paper, $p = 0$ yields the case of (103), and $q = 0$, $p = 3$ yields the standard Φ^3 field theory, while $q = 0$, $p = 4$ yields the standard Φ^4 field theory. The local stability condition for the second case has been derived in [10] and the existence of stable bound-state solutions in the latter two cases has been ruled out already by Derrick's theorem. We now generalize Derrick's theorem to the class of theories characterized by (105)–(108) for the case of time-dependent but stationary bound-state solutions of the form $\Psi_j(r,t) = \psi_j(r)\mathrm{e}^{-iE_jt}$, $\Phi(r,t) = \phi(r)$ and for the case of arbitrary p and q. Since this class is less general than in the case of Derrick's theorem, we will benefit by being able to derive two additional virial relations.

For stationary fields the equations of motion (106), (107) and the normalization condition (108) become respectively

$$(\Delta - \gamma_j^2 + gq\phi^p(\psi_j^*\psi_j)^{q-1})\psi_j = 0 \tag{109}$$

$$(\Delta - \mu_0^2)\phi = -gp\phi^{p-1}\sum_{i=1}^{N}(\psi_i^*\psi_i)^q \tag{110}$$

and

$$\langle\psi_j|\psi_j\rangle \equiv 2|E_j|\int d^3r\,|\psi_j(r)|^2 = 1 \qquad j = 1,\ldots,N \tag{111}$$

where $\gamma_j^2 = m_0^2 - E_j^2 > 0$. The corresponding total energy is

$$H = \int d^3r\Big(\sum_{j=1}^{N}[\nabla\psi_j^*\cdot\nabla\psi_j + (E_j^2+m_0^2)\psi_j^*\psi_j - g\phi^p(\psi_j^*\psi_j)^q] + \tfrac{1}{2}[(\nabla\phi)^2+\mu_0^2\phi^2]\Big)$$
$$\equiv H_1 + H_2 - H_I + H_3 + H_4. \tag{112}$$

Notice that (111) implies

$$H_2 = \sum_{j=1}^{N}\left[|E_j| + \frac{m_0}{2}\left(\frac{m_0}{|E_j|} - \frac{|E_j|}{m_0}\right)\right]. \tag{113}$$

In order to make the variation of the energy δH we have to choose a proper energy functional. Equations (111) tell us that we are dealing with a constrained system. According to the standard rules of quantum mechanics there are two ways to perform variation of the energy δH for a constrained system. The first [11] is to use an energy functional which does not depend on the norm of the variational functions corresponding to ψ_j. The second [12] is to introduce the normalization condition by real Lagrange multipliers. Both ways are equivalent, but for our purposes the first is more convenient. The unique functional of the variational fields $\psi_{\lambda j}$ and ϕ_λ, which fulfils the above requirement and which reduces to (112) in the case $\psi_{\lambda_j} = \psi_j$, $\phi_\lambda = \phi$ is

$$H(\lambda) = \int d^3r\sum_{j=1}^{N}\left[\frac{\nabla\psi_{\lambda j}^*\cdot\nabla\psi_{\lambda j} + (E_j^2+m_0^2)\psi_{\lambda j}^*\psi_{\lambda j}}{\langle\psi_{\lambda_j}|\psi_{\lambda j}\rangle} - g\phi_\lambda^p\frac{(\psi_{\lambda j}^*\psi_{\lambda j})^q}{\rangle\psi_{\lambda j}|\psi_{\lambda j}\rangle^q}\right]$$
$$+\tfrac{1}{2}\int d^3r[(\nabla\phi_\lambda)^2+\mu_0^2\phi_\lambda^2]. \tag{114}$$

Now to perform the variation δH we have to choose a set of variational fields $\psi_{\lambda j}$ and ϕ_λ, which is large enough to yield all the solutions of $\delta H = 0$ or, equivalently, the equations of motion (109) and (110). A suitable set of variational fields is

$$\psi_{\lambda j}(r) = \lambda^t\psi(\lambda r) \qquad \psi_{\lambda j}^*(r) = \lambda^{t^*}\psi^*(\lambda r) \qquad \phi_\lambda(r) = \lambda^s\phi(\lambda r) \tag{115}$$

where t is an arbitrary complex and λ and s are arbitrary real numbers. The variational fields $\psi_{\lambda j}$ are not normalized except for $\lambda = 1$ in which case the normalization is defined in (111). For $\lambda = 1$ we have

$$\psi_{1j}(r) = \psi_j(r) \qquad \psi_{1j}^*(r) = \psi_j^*(r) \qquad \phi_1(r) = \phi(r) \qquad H(1) = H. \tag{116}$$

The variations $\delta\psi_j$, $\delta\psi_j^*$, $\delta\phi$, δH and $\delta^2 H$ are defined by

$$\delta\psi \equiv \left(\frac{d\psi_{\lambda j}}{d\lambda}\right)_{\lambda=1} = t\psi(r) + r\cdot\nabla\psi(r) \qquad \delta\psi^* \equiv \left(\frac{d\psi_{\lambda j}^*}{d\lambda}\right)_{\lambda=1} = t^*\psi^*(r) + r\cdot\nabla\psi^*(r) \tag{117}$$

and

$$\delta\phi \equiv \left(\frac{\mathrm{d}\phi_\lambda}{\mathrm{d}\lambda}\right)_{\lambda=1} = s\phi(r) + r\cdot\nabla\phi(r) \qquad \delta H = \left(\frac{\mathrm{d}H(\lambda)}{\mathrm{d}\lambda}\right)_{\lambda=1}$$

$$\delta^2 H = \left(\frac{\mathrm{d}^2 H(\lambda)}{\mathrm{d}\lambda^2}\right)_{\lambda=1}. \tag{118}$$

Notice that at each r the variations $\delta\psi_j$, $\delta\psi_j^*$ and $\delta\phi$ are arbitrary and independent of each other. Therefore, applying the standard variational procedure of quantum mechanics, we conclude that $\delta H = 0$ is equivalent to (111)–(112).

Since the integration in (114) is over the entire space, we can eliminate the dependence of the fields on λr by making a suitable rescaling of the integration variable and obtain

$$H(\lambda) = \lambda^2 H_1 + H_2 - \lambda^{sp+3(q-1)} H_I + \lambda^{2s-1} H_3 + \lambda^{2s-3} H_4 \tag{119}$$

where $H_1, \ldots, H_4$ and H_I, which are independent of λ, are defined in (112). From (119) and $\delta H = 0$ it follows that

$$2H_1 - [sp + 3(q-1)]H_I + (2s-1)H_3 + (2s-3)H_4 = 0. \tag{120}$$

Since (120) must be valid for all s, it implies two separate virial relations

$$2H_3 + 2H_4 - pH_I = 0 \tag{121}$$

and

$$2H_1 - H_3 - 3H_4 - 3(q-1)H_I = 0. \tag{122}$$

Moreover, multiplying (109) by ψ_j^*, integrating over the entire space and summing over all $j = 1, \ldots, N$, we obtain a third independent virial relation

$$H_1 + H_2 - qH_I = \sum_{j=1}^{N} |E_j|. \tag{123}$$

Combining these relations one can obtain other useful relations. For instance, eliminating H_1 and H_I and using (113) we obtain

$$[2(3-q) - p]H_3 + [2(3-q) - 3p]H_4 = m_0 \sum_{j=1}^{N} \left(\frac{m_0}{|E_j|} - \frac{|E_j|}{m_0}\right). \tag{124}$$

Since H_3, H_4 and the right-hand side of (124) are positive numbers, we obtain a necessary condition for the existence of bound states

$$2(3-q) - p > 0. \tag{125}$$

Another useful relation is obtained if we eliminate H_3 from (124) by means of (121)

$$\frac{p}{2}[2(3-q) - p]H_I = m_0 \sum_{j=1}^{N} \left(\frac{m_0}{|E_j|} - \frac{|E_j|}{m_0}\right) + 2pH_4. \tag{126}$$

This is the equation (61) which we used above to determine the mass.

Now we compute the second variation of the energy $\delta^2 H$ and derive the local stability conditions which ensure that $\delta^2 H > 0$ for all values of s and t, which are the parameters spanning the set of variations. Equation (119) implies

$$\delta^2 H = 2H_1 - [sp+3(q-1)][sp+3(q-1)-1]H_I + (2s-1)(2s-2)H_3 + (2s-3)(2s-4)H_4. \tag{127}$$

Eliminating H_1, H_3 and H_I by means of the virial relations (121), (122) and (124), we obtain

$$\begin{aligned}\delta^2 H = {} & \frac{2p(2-p)s^2 - 4p(3q-2)s + 3p - 6(q-1)(3q-5)}{2(3-q)-p} m_0 \sum_{j=1}^{N} \left(\frac{m_0}{|E_j|} - \frac{|E_j|}{m_0} \right) \\ & + \{[2p(2-p)s^2 - \{12p(q-1) + 8(3-q)\}s - 3p + 12(3-q) \\ & - 6(q-1)(3q-5)]/(2(3-q)-p)\}2H_4. \end{aligned} \tag{128}$$

Consider the case when $\mu_0 = 0$. From (112) we have $H_4 = 0$, and hence (128) implies that $\delta^2 H > 0$ is equivalent to

$$2p(2-p)s^2 - 4p(3q-2)s + 3p - 6(q-1)(3q-5) > 0 \qquad \mu_0 = 0. \tag{129}$$

The solution of this inequality is

$$\tfrac{1}{3}(4-p) - \tfrac{1}{6}|2-p| < q < \tfrac{1}{3}(4-p) + \tfrac{1}{6}|2-p| \qquad \mu_0 = 0. \tag{130}$$

Notice that for $p = 0$ (130) coincides with the local stability condition proved in [10], which we quoted in equation (103). Also notice that the only positive integers p and q which can satisfy (130) are $p = 1$ and $q = 1$, which is the case investigated in this paper. To have a sensible field theory p and q must be positive integers. Therefore the result—that stability alone restricts the choice among the class of theories defined by (105) to just one case: $p = q = 1$—must be considered as satisfactory.

For each p and q obeying (130), the local stability for a sufficiently small $\mu_0 \neq 0$ follows by continuity of H_4. The local stability condition $\delta^2 H > 0$ implies an upper bound on μ_0. For instance, for $p = 1$ and $q = 1$ we substitute equation (75) in (128), which in the present notation reads

$$H_4 = \beta m_0 N \left(\frac{m_0}{|E|} - \frac{|E|}{m_0} \right) \tag{131}$$

and obtain the local stability condition for the ground state

$$s^2 - 2\frac{1+8\beta}{1+2\beta}s + \frac{3}{2}\frac{1+14\beta}{1+2\beta} > 0 \tag{132}$$

which must be satisfied for all real s. Solving for β, we obtain

$$\beta < \frac{2}{11}\left(1 + \frac{3\sqrt{3}}{4}\right) \tag{133}$$

and then using (77) and (79)

$$\sigma < 1.0 \qquad \mu_0 < \sqrt{2} m_0. \tag{134}$$

Thus a locally stable ground state for the theory investigated in this paper ($p = q = 1$) exists only if $\sigma < 1$.

Acknowledgments

Most of this work was done at the Weizmann Institute of Science, Rehovot, Israel. The author wishes to thank Y Eisenberg, S A Gurvitz and D Kazhdan for valuable discussions and suggestions.

Appendix

In this appendix we develop a simple algorithm for the numerical determination of the functions needed for the evaluation of the functions R_σ and $\mathcal{V}_\sigma$. As a first step we split the recurrence relation (32) into three separate parts to avoid multiple integration

$$\varphi_{nm}(y,\sigma) = \int_0^y \mathrm{d}t\, \phi_{nm}(t,\sigma) \tag{135}$$

$$\phi_{nm}(y,\sigma) = \frac{1}{(y+n\sigma+m+\frac{1}{2})^2-\frac{1}{4}}\Bigg(\varphi_{n-1,m}(y,\sigma) + \sum_{k=0}^{n}\sum_{l=1}^{m}\int_0^y \mathrm{d}t\varphi_{n-k,m-l}(y-t,\sigma)f_{kl}(t,\sigma)\Bigg) \tag{136}$$

and

$$f_{nm}(y,\sigma) = \frac{1}{(y+n\sigma+m)^2-\sigma^2}\Bigg(\varphi_{n,m-1}(y,\sigma) + \sum_{k=0}^{n}\sum_{l=1}^{m}\int_0^y \mathrm{d}t\varphi_{n-k,m-l}(y-t,\sigma)\phi_{k,l-1}(t,\sigma)\Bigg). \tag{137}$$

It is easy to see that these equations are explicit recurrence relations, rather than integral equations. In order to convert these recurrence relations to a form digestible by computers, we divide the interval $[0, y]$ into $\mathcal{N}$ pieces each of length x and use the trapezoidal rule to evaluate the integrals. As a result of the discretization we obtain a new set of functions $\overline{\varphi}_{nm}(x_j,\sigma)$, $\overline{\phi}_{nm}(x_j,\sigma)$ and $\overline{f}_{nm}(x_j,\sigma)$ defined on the grid of points $x_0 \equiv 0, x_1, \ldots, x_{\mathcal{N}} \equiv y$, which converge to the true functions $\varphi_{nm}(x_j,\sigma)$, $\phi_{nm}(x_j,\sigma)$ and $f_{nm}(x_j,\sigma)$ as $x \to 0$. The corresponding recurrence relations are

$$\overline{\varphi}_{nm}(x_j,\sigma) = \overline{\varphi}_{nm}(x_{j-1},\sigma) + \tfrac{1}{2}x(\overline{\phi}_{nm}(x_{j-1},\sigma) + \overline{\phi}_{nm}(x_j,\sigma)) \tag{138}$$

$$\overline{\phi}_{nm}(x_j,\sigma) = \frac{1}{(x_j+n\sigma+m+\frac{1}{2})^2-\frac{1}{4}}\Bigg(\overline{\varphi}_{n-1,m}(x_j,\sigma) + \frac{x}{2}\overline{f}_{nm}(x_j,\sigma) + \frac{x}{2}\frac{\overline{\varphi}_{n,m-1}(x_j,\sigma)}{1-\sigma^2} + x\sum_{k=0}^{n}\sum_{l=1}^{m}\sum_{i=1}^{j-1}\overline{\varphi}_{n-k,m-l}(x_j-x_i,\sigma)\overline{f}_{kl}(x_i,\sigma)\Bigg) \tag{139}$$

and

$$\overline{f}_{nm}(x_j,\sigma) = \frac{1}{(x_j+n\sigma+m)^2-\sigma^2}\Bigg(\overline{\varphi}_{n,m-1}(x_j,\sigma) + \frac{x}{2}\overline{\phi}_{n,m-1}(x_j,\sigma) + \frac{x}{2}\frac{\overline{\varphi}_{n-1,m-1}(x_j,\sigma)}{\sigma(\sigma+1)} + x\sum_{k=0}^{n}\sum_{l=1}^{m}\sum_{i=1}^{j-1}\overline{\varphi}_{n-k,m-l}(x_j-x_i,\sigma)\overline{\phi}_{k,l-1}(x_i,\sigma)\Bigg) \tag{140}$$

where $x_j = jx$ and $j = 1, \ldots, \mathcal{N}$, $n = 0, 1, \ldots$, $m = 0, 1, \ldots$. The initial data for the recurrence process are

$$\overline{\varphi}_{-1,m}(x_j, \sigma) = 0 \qquad \overline{\varphi}_{n,-1}(x_j, \sigma) = 0 \qquad \overline{\varphi}_{nm}(0, \sigma) = 0 \tag{141}$$

except for $n = m = 0$ in which case

$$\overline{\varphi}_{00}(0, \sigma) = 1 \tag{142}$$

and

$$\overline{\phi}_{n,-1}(x_j, \sigma) = 0 \qquad \overline{\phi}_{nm}(0, \sigma) = 0 \tag{143}$$

except for $n = 1$, $m = 0$ in which case

$$\overline{\phi}_{10}(0, \sigma) = \frac{1}{\sigma(\sigma + 1)}. \tag{144}$$

The recurrence process starts with the evaluation of the cycle (140) → (139) → (138) for $n = m = 0$ and $j = 1$. Then n, m and j are iterated until certain maximal values, $j_{\max}$, $n_{\max}$ and $m_{\max}$ say, are reached, which are determined by the step function in (31). The result of the recurrence process is $\overline{\varphi}_{nm}(x_1, \sigma), \ldots, \overline{\varphi}_{nm}(x_{j_{\max}}, \sigma)$, $n = 0, \ldots, n_{\max}$, $m = 0, \ldots, m_{\max}$. In order to obtain an estimate of the difference between $\overline{\varphi}_{nm}(x_j, \sigma)$ and $\varphi_{nm}(x_j, \sigma)$, the grid is refined by replacing $\mathcal{N}$ by $2\mathcal{N}$ and x by $x/2$, and the recurrence process is repeated. The resulting $\overline{\varphi}_{nm}(x_2, \sigma), \overline{\varphi}_{nm}(x_4, \sigma), \ldots$ are compared with the previously calculated $\overline{\varphi}_{nm}(x_1, \sigma), \overline{\varphi}_{nm}(x_2, \sigma), \ldots$. The process terminates when a certain specified accuracy is reached and the $\overline{\varphi}_{nm}(x_1, \sigma), \overline{\varphi}_{nm}(x_2, \sigma), \ldots$ can be considered to be identical to $\varphi_{nm}(x_1, \sigma), \varphi_{nm}(x_2, \sigma), \ldots$. Notice that the numerical evaluation of the functions φ_{nm} does not mean that we are solving the problem numerically. Rather, it means that the solution (31) is given in terms of non-standard functions and that we have to teach our computer to obtain the values of these well defined analytical functions.

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